The notion of limit of a function:

The intuitive idea of a function f having a limit l at the point 'c' is that the values $f(x)$ are close to l when x is close to (but different from) c. It is immaterial whether f is defined at c or not.

Definition:-

Let $f(x)$ be defined on a open interval about c, except possibly at c itself. We say that the limit of $f(x)$ as x approaches c is the number l, and write

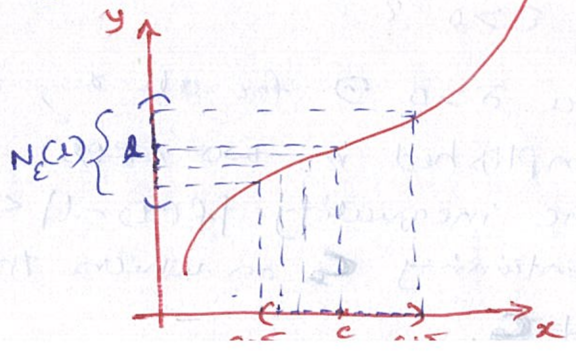
$$\lim_{x \to c} f(x) = l,$$

if, for every no. $\varepsilon > 0$ $\exists$ a corresponding no. $\delta > 0$ such that for all x,

$$0 < |x - c| < \delta \Rightarrow |f(x) - l| < \varepsilon.$$

Note:- 1. The formal definition of limit does not tell us how to find the limit of a function, but it enables us to verify that a suspected limit is correct.

2. If the limit of f at c does not exist, we say that f diverges at c



Examples:-

1. Show that $\lim_{x \to 1} (5x-3) = 2$, using ε-δ def. of limit.

Sol:- Set $c = 1$, $f(x) = 5x-3$, and $l = 2$.

For a given $\varepsilon > 0$, we have to find a suitable $\delta > 0$ so that if $x \neq 1$ and x is within distance δ of $c = 1$, i.e., whenever

$$0 < |x-1| < \delta,$$

it is true that $f(x)$ is within distance ε of $l = 2$, so

$$|f(x) - 2| < \varepsilon.$$

We find δ by working backward from the ε-inequality

$$|(5x-3)-2| = |5x-5| = 5|x-1| < \varepsilon.$$

$$\Rightarrow |x-1| < \varepsilon/5.$$

Thus, we can take $\delta = \varepsilon/5$.

If $0 < |x-1| < \delta = \varepsilon/5$, then $|f(x) - 2|$

$$= |(5x-3)-2|$$

$$= 5|x-1| < 5.\delta = 5.\varepsilon/5$$

$\therefore |f(x) - 2| < \varepsilon$.

Note:- Any smaller positive $\delta \ (< \varepsilon/5)$ will do as well.

REMARK: "How to find algebraically a δ for a given f, l, c and $\varepsilon > 0$"?

Finding a $\delta > 0 \ni$ for all x, $0 < |x-c| < \delta \Rightarrow |f(x) - l| < \varepsilon$ can be accomplished in two steps:

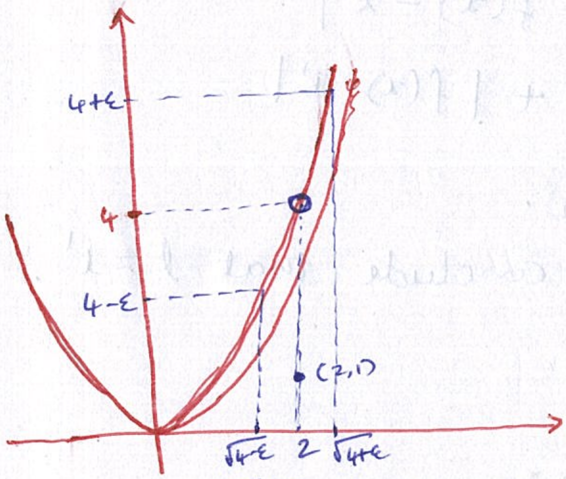
(i) Solve the inequality $|f(x) - l| < \varepsilon$ to find an open interval (a, b) containing c on which the inequality holds for all $x \neq c$.

(ii) Find a value of $\delta > 0$ that places the open interval $(c-\delta, c+\delta)$ centered at c inside the interval (a,b). The inequality $|f(x) - l| < \varepsilon$ will hold for all $x \neq c$ in this δ-interval.

Example:-

1. Prove that $\lim_{x \to 2} f(x) = 4$ if $f(x) = \begin{cases} x^2, & x \neq 2 \\ 1, & x = 2. \end{cases}$ using $\varepsilon - \delta$ def.

Sol:- (i) for $x \neq c (=2)$, we have $f(x) = x^2$, and the inequality to solve is $|f(x) - 4| < \varepsilon$



ie, $|x^2 - 4| < \varepsilon$.

$\Rightarrow -\varepsilon < x^2 - 4 < \varepsilon$

$\Rightarrow 4 - \varepsilon < x^2 < 4 + \varepsilon$

$\Rightarrow \sqrt{4-\varepsilon} < |x| < \sqrt{4+\varepsilon}$

$\Rightarrow \sqrt{4-\varepsilon} < x < \sqrt{4+\varepsilon}$

The inequality $|f(x) - 4| < \varepsilon$ holds for all $x \neq 2$ in the open interval $(\sqrt{4-\varepsilon}, \sqrt{4+\varepsilon})$.

(ii) choose $\delta > 0 \ni (2-\delta, 2+\delta) \subseteq (\sqrt{4-\varepsilon}, \sqrt{4+\varepsilon})$.

In other words, take $\delta = \min\{2 - \sqrt{4-\varepsilon}, \sqrt{4+\varepsilon} - 2\}$.

$\therefore$ For any given $\varepsilon > 0$, choose $\delta = \min\{2 - \sqrt{4-\varepsilon}, \sqrt{4+\varepsilon} - 2\}$ then for all $x \ni 0 < |x - c| < \delta \Rightarrow |f(x) - l| < \varepsilon$.

Result: If $f(x)$ is defined on an open interval (D) about c, except possibly at c itself, then f has atmost one limit at c. i.e, if limit exists, then that limit is unique.

Proof: Suppose l and l' are limits of f at c.

For any $\varepsilon > 0$ $\exists$ $\delta(\varepsilon/2) > 0$ $\ni$ if $x \in D$ &

$$0 < |x-c| < \delta, \text{ then } |f(x) - l| < \varepsilon/2.$$

Also for this $\varepsilon > 0$ $\exists$ $\delta'(\varepsilon/2) > 0$ $\ni$ if $x \in D$ &

$$0 < |x-c| < \delta', \text{ then } |f(x) - l'| < \varepsilon/2$$

Consider
$$\begin{aligned} |l - l'| &\le |l - f(x) + f(x) - l'| \\ &\le |l - f(x)| + |f(x) - l'| \\ &< \varepsilon/2 + \varepsilon/2. \end{aligned}$$

$\because$ $\varepsilon > 0$ is arbitrary, we conclude that $l = l'$.

*

Sequential Criterion:

The following are equivalent statements:

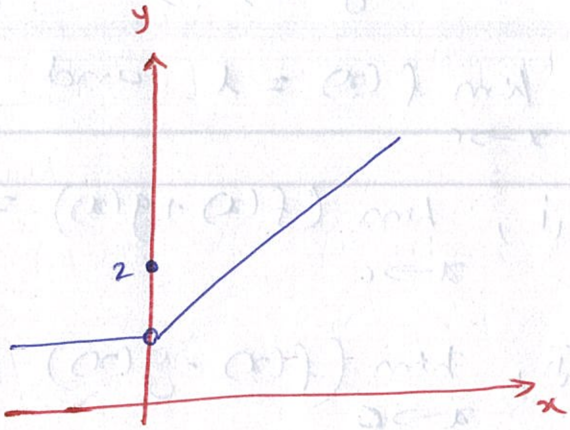
i, $\lim_{x \to c} f = l$

ii, For every seq. $\langle x_n \rangle$ in the domain of f that converges to c $\ni$ $x_n \neq c$ $\forall n \in \mathbb{N}$, the sequence $\langle f(x_n) \rangle$ converges to l.

Examples:-

i) Consider $f: \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 1 & \text{if } x < 0 \\ 2 & \text{if } x = 0 \\ x+1 & \text{if } x > 0 \end{cases}.$$



Then $\lim_{x \to 0} f(x) = 1$.

Sol:- If $\langle x_n \rangle$ is a sequence in $\mathbb{R} \setminus \{0\}$ s.t $x_n \to 0$, then

$$f(x_n) = \begin{cases} 1 & \text{if } x_n < 0 \\ x_n + 1 & \text{if } x_n > 0 \end{cases},$$

and hence by seq. criterion, we can see that $f(x_n) \to 1$.

ii) If $f: \mathbb{R} - \{0\} \to \mathbb{R}$ is defined by $f(x) = 1/x$, then $\lim_{x \to 0} (1/x)$ does not exist in $\mathbb{R}$.

Sol:- If we take the seq. $\langle x_n \rangle$ with $x_n = 1/n$ for $n \in \mathbb{N}$, then $\lim \langle x_n \rangle = 0$, but $f(x_n) = f(1/n) = n$, hence $\langle f(x_n) \rangle = \langle n \rangle$ doesn't converge in $\mathbb{R}$, since it is not bounded.

Hence by sequential criterion, $\lim_{x \to 0} (1/x)$ does not exist.

Basic Properties:-

If l, m, c and r are real numbers and

$\lim_{x \to c} f(x) = l$ and $\lim_{x \to c} g(x) = m$, then

i, $\lim_{x \to c} (f(x) + g(x)) = l + m$

ii, $\lim_{x \to c} \left(f(x) - g(x)\right) = l - m$

iii, $\lim_{x \to c} (r \cdot f(x)) = r \cdot l$

iv, $\lim_{x \to c} \left(f(x) \cdot g(x)\right) = l \cdot m$

v, If $m \neq 0$, $g(x) \neq 0 \;\; \forall x \in D$omain, then

$$\lim_{x \to c} \left(\frac{f}{g}\right)(x) = \frac{l}{m}.$$

vi, If $f(x) \geq 0 \;\; \forall x \in$ Domain of f, then $l \geq 0$ and

$\lim_{x \to c} \sqrt[n]{f(x)} = \sqrt[n]{l} = l^{1/n}$, for any $n \in \mathbb{N}$.

Consequences:

i, If P is a polynomial function, i.e.,

$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0$, then

$$\lim_{x \to c} P(x) = P(c) = a_n c^n + a_{n-1} c^{n-1} + \cdots + a_0.$$

ii, If P and q are polynomial functions on $\mathbb{R}$ and if $q(c) \neq 0$, then $\lim_{x \to c} \frac{P(x)}{q(x)} = \frac{P(c)}{q(c)}$.

(iii) [The Sandwich Theorem / The Squeeze Theorem]

Suppose that $g(x) \le f(x) \le h(x)$ for all x in some open interval containing c, except possibly at $x=c$ itself. Suppose also that

$$\lim_{x\to c} g(x) = \lim_{x\to c} h(x) = l.$$

Then $\lim_{x\to c} f(x) = l$.

Examples:

1. Given that

$$1-\frac{x^2}{4} \le f(x) \le 1+\frac{x^2}{2} \quad \forall\, x \ne 0,$$

no matter how complicated f is, find $\lim_{x\to 0} f(x)$,

Sol:- Since $\lim_{x\to 0}\left(1-\left(\frac{x^2}{4}\right)\right) = 1$ and $\lim_{x\to 0}\left(1+\left(\frac{x^2}{2}\right)\right) = 1$,

by Sandwich theorem, $\lim_{x\to 0} f(x) = 1$.

2. $\lim_{x\to 0}\left(\frac{\cos x - 1}{x}\right) = 0$.

Sol:- Observe that $1-\frac{1}{2}x^2 \le \cos x \le 1 \quad \forall\, x \in \mathbb{R}$.

[Prove this inequality as an Ex.]

$$\Rightarrow -\frac{1}{2}x \le \frac{\cos x - 1}{x} \le 0 \quad \text{for } x>0$$

$$\& \quad 0 \le \frac{\cos x - 1}{x} \le -\frac{1}{2}x \quad \text{for } x<0.$$

Now let $f(x) = -\frac{x}{2}$ for $x \ge 0$ & $f(x) = 0$ for $x<0$

& let $h(x) = 0$ for $x \ge 0$ & $h(x) = -\frac{x}{2}$ for $x<0$.

Then we have, $f(x) \le \frac{\cos x - 1}{x} \le h(x)$ for $x \ne 0$.

Since $\lim_{x \to 0} f(x) = 0 = \lim_{x \to 0} h(x)$, by squeeze th.

we have $\lim_{x \to 0} \frac{\cos x - 1}{x} = 0$.

One-Sided Limits:

Def:- If $f(x)$ is defined on the interval (c, b), where $c < b$, and approaches arbitrarily close to l as x approaches c from within that interval, then f has "right-hand limit" l at c.

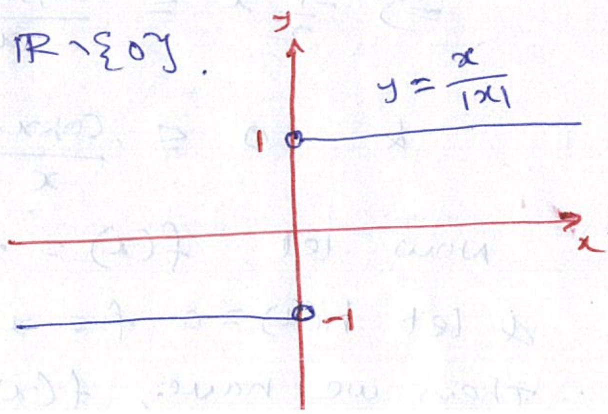
we write, $\lim_{x \to c^+} f(x) = l$.

Similarly, if $f(x)$ is defined on an interval (a, c), where $a < c$ and approaches arbitrarily close to m as x approaches c from within that interval, then f has "left-hand limit" m at c.

we write, $\lim_{x \to c^-} f(x) = m$.

Fact:- $\lim_{x \to c} f(x)$ exists $\iff$ $\lim_{x \to c^+} f(x)$ & $\lim_{x \to c^-} f(x)$ exists and are equal.

Example:- $f(x) = \frac{x}{|x|}$ $\forall x \in \mathbb{R} \setminus \{0\}$.



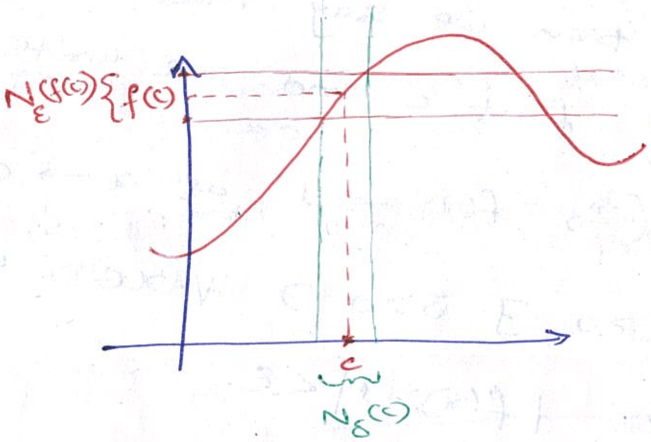
$\lim_{x \to 0^+} f(x) = 1$ &

$\lim_{x \to 0^-} f(x) = -1$.

Continuity:

Intuitively, a continuous real valued function of a real variable is a function whose graph is "unbroken" ie, it can be sketched in a "continuous" manner.

Def:- Let $D \subseteq \mathbb{R}$, and $f: D \to \mathbb{R}$ be a function. Let $c \in D$. We say that f is continuous at c if, given any $\varepsilon > 0$ $\exists$ $\delta > 0$ $\ni$ if x is any point of D satisfying $|x - c| < \delta \Rightarrow |f(x) - f(c)| < \varepsilon$



Given $N_\varepsilon(f(c))$, a neighborhood $N_\delta(c)$ is to be determined.

Note: 1. If f fails to be continuous at c, then we say that f is discontinuous at c.

2. If $c \in D$ is a cluster point of D, then f is continuous at c "if and only if" $f(c) = \lim_{x \to c} f(x)$

3. If $c \in D$ is a cluster point of D, then three conditions must hold for f to be continuous at c:
   (i) f must be defined at c
   (ii) the limit of f at c must exist in $\mathbb{R}$.
   (iii) these two values must be equal.

Sequential Criterion for Continuity:

A function $f: D \to \mathbb{R}$ is continuous at the point $c \in D$ $\iff$ for every sequence $\langle x_n \rangle$ in D that converges to c, the sequence $\langle f(x_n) \rangle$ converges to $f(c)$.

Def:- Let $D \subseteq \mathbb{R}$ and let $f: D \to \mathbb{R}$. If $C \subseteq D$, we say that f is continuous on the set C if f is continuous at every point of C.

Note! f is said to be discontinuous if f is not continuous.

Basic Properties!

1. Let $f, g: D \to \mathbb{R}$ be functions, where $D \subseteq \mathbb{R}$ and let $c \in \mathbb{R}$.
Suppose f and g are continuous at c. Then
   a) $f+g$, $f-g$, fg, and rf (for any $r \in \mathbb{R}$) are continuous at c.
   b) Further if $g(x) \neq 0$ $\forall$ $x \in D$ & g is continuous at c, then f/g is continuous at c.

2. Let $D \subseteq \mathbb{R}$, $f: D \to \mathbb{R}$, and let $|f|$ be defined by $|f|(x) = |f(x)|$ for $x \in D$.
   a) If f is continuous at a pt. $c \in D$, then $|f|$ is continuous at c.
   b) If f is continuous on D, then $|f|$ is continuous on D.
   c) If $f(x) \geq 0$ $\forall$ $x \in D$ & f is continuous at c, then $\sqrt{f}$ is continuous at c.

3. If f is continuous at c and $f(c) \neq 0$, then $\exists\ \delta > 0 \ni f(x)$ and $f(c)$ have the same sign.
[and in particular, $f(x) \neq 0\ \forall\ x \in N_\delta(c) \cap D$]

4. [Composites of continuous functions]

Let $D, E \subseteq \mathbb{R}$ and let $f: D \to \mathbb{R}$, $g: E \to \mathbb{R}$ be functions such that $f(D) \subseteq E$.

If f is continuous at a pt. $c \in D$ & g is continuous at $b = f(c) \in E$, then the composition $g \circ f : D \to \mathbb{R}$ is continuous at c.

Examples of continuous functions:

1. Polynomial functions are continuous everywhere.

2. The n^{th} root function $x \mapsto \sqrt[n]{x}$ is continuous on $[0, \infty)$.

3. The absolute value function $x \mapsto |x|$ is continuous on $\mathbb{R}$.

4. The signum function, $\operatorname{sgn}(x) = \begin{cases} +1 & \text{for } x > 0 \\ 0 & \text{for } x = 0 \\ -1 & \text{for } x < 0 \end{cases}$

is not continuous at 0.

5. The Dirichlet function $f: \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$$

is discontinuous at every $x \in \mathbb{R}$.

6. Rational functions.

If p and q are polynomial functions on $\mathbb{R}$, then there are at most a finite number $\alpha_1, \dots, \alpha_m$ of real roots of q. If $x \notin \{\alpha_1, \dots, \alpha_m\}$ then $q(x) \neq 0$ so that we can define the rational function r by

$$\gamma(x) = \frac{P(x)}{q(x)} \quad \text{for } x \notin \{\alpha_1, \ldots, \alpha_m\}.$$

It was seen earlier that if $q(c) \neq 0$, then

$$\gamma(c) = \frac{P(c)}{q(c)} = \lim_{x \to c} \frac{P(x)}{q(x)} = \lim_{x \to c} \gamma(x).$$

In other words γ is cont. at c. $\because$ c is any real no. that is not a root of q, we say that "a rational fn. is continuous at every real number for which it is defined".

Boundedness of continuous fn's on closed & bounded intervals:

Theorem:-

If $f : [a,b] \to \mathbb{R}$ is a continuous fn. on a closed and bounded interval, then f is bounded and attains its bounds. i.e. both $\sup\{f(x) \mid x \in [a,b]\}$ and $\inf\{f(x) \mid x \in [a,b]\}$ exist. To be particular, $\exists$ pts $x^* \in [a,b]$, $x_* \in [a,b]$ $\ni$

$$f(x^*) = \sup\{f(x) \mid x \in [a,b]\} \quad \&$$

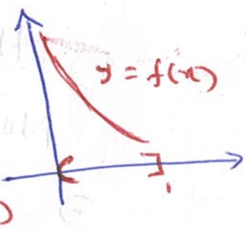
$$f(x_*) = \inf\{f(x) \mid x \in [a,b]\}.$$

* We say f has absolute maximum on $[a,b]$ at x^* & f has absolute minimum at x_*.

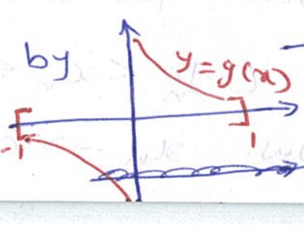
Note:- The result may not hold if the domain isn't a closed and bounded interval or if the function is not continuous. For example,

i) Consider $f(x) = \frac{1}{x}$, where $f : (0,1] \to \mathbb{R}$.

f is cont but f is not bounded.



ii) Consider $g : [-1,1] \to \mathbb{R}$ defined by



$$g(x) = \begin{cases} 1/x & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

iii, Consider $h: \mathbb{R} \to \mathbb{R}$ by $h(x) = \frac{1}{1+x^2}$

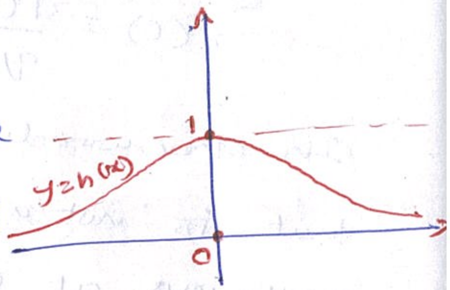
h is continuous on $\mathbb{R}$. Note that h is bounded.

Indeed, $0 < h(x) \le 1 \; \forall x \in \mathbb{R}$,

but h doesn't attain its bounds since

$\inf\{f(x) \mid x \in \mathbb{R}\} = 0 \neq f(c)$ for any

$c \in \mathbb{R}$.



[Note: Here the domain is unbounded interval]

Bolzano's Intermediate value theorem: (IVT)

Note:- It assures us that a continuous fn. on an interval takes on any number that lies between two of its values.

Theorem:- If I is an interval in $\mathbb{R}$ and $f: I \to \mathbb{R}$ is continuous, then f has the "Intermediate value Property" (IVP) on I. ie, If $a, b \in I$ and if $k \in \mathbb{R}$ satisfies $f(a) < k < f(b)$ then $\exists$ a pt. $c \in I$ between a and b $\ni f(c) = k$.

*Here k is intermediate value

Proof:- Suppose that $a < b$ and let $g(x) = f(x) - k$

then $g(a) < 0 < g(b)$.

Consider $S = \{x \in [a,b] \mid g(x) \le 0\}$.

Then $S \neq \phi$ ($\because a \in S$) and S is bounded above.
(b is an u.b).

Hence $\sup S$ exists. Let $c = \sup S$.

Now for each $n \in \mathbb{N}$, $c - \frac{1}{n} < c$ & so $\exists\, x_n \in S$

$\ni c - \frac{1}{n} < x_n \le c$.

Now $x_n \to c$ as $n \to \infty$. So by continuity $g(x_n) \to g(c)$.

$\because g(x_n) \le 0 \quad \forall x_n \in S$

$\Rightarrow g(c) \le 0.$

On the other hand, $c + \frac{1}{n} \notin S$, and hence

$g(c + \frac{1}{n}) > 0 \quad \forall n$

By Continuity, $g(c) \ge 0$.

$\therefore g(c) = 0 \Rightarrow f(c) - k = 0 \Rightarrow f(c) = k.$

~~The~~

~~Examples:~~

Remarks:- [Consequences of Intermediate value theorem]

1. The graph of a function continuous on an interval cannot have any breaks over the interval. The curve will be "connected".

2. We call a solution of the equation $f(x) = 0$ a "root" of the equation or "zero" of the fn. f. The intermediate value theorem tells us that if f is continuous, then any interval on which f changes sign contains a zero of the function.

Example:-

Use I.V.T to prove that the equation

$$\sqrt{2x+5} = 4 - x^2$$

has a solution.

Sol:- We rewrite the equation as

$$\sqrt{2x+5} + x^2 = 4,$$

and set $f(x) = \sqrt{2x+5} + x^2$.

Now $g(x) = \sqrt{2x+5}$ is continuous on the interval $[-5/2, \infty)$.

Also $f(x) = \sqrt{2x+5} + x^2$ is continuous on the interval $[-5/2, \infty)$.

By trial and error, we find the fn. value $f(0) = \sqrt{5} \approx 2.24$ and $f(2) = \sqrt{9} + 4 = 7$, and note that f is also continuous on the finite closed interval $[0,2] \subset [-5/2, \infty)$.

Since the value $y_0 = 4$ is between the no's 2.24 and 7, by the I.V.T, $\exists\ c \in [0,2] \ni f(c) = 4$.